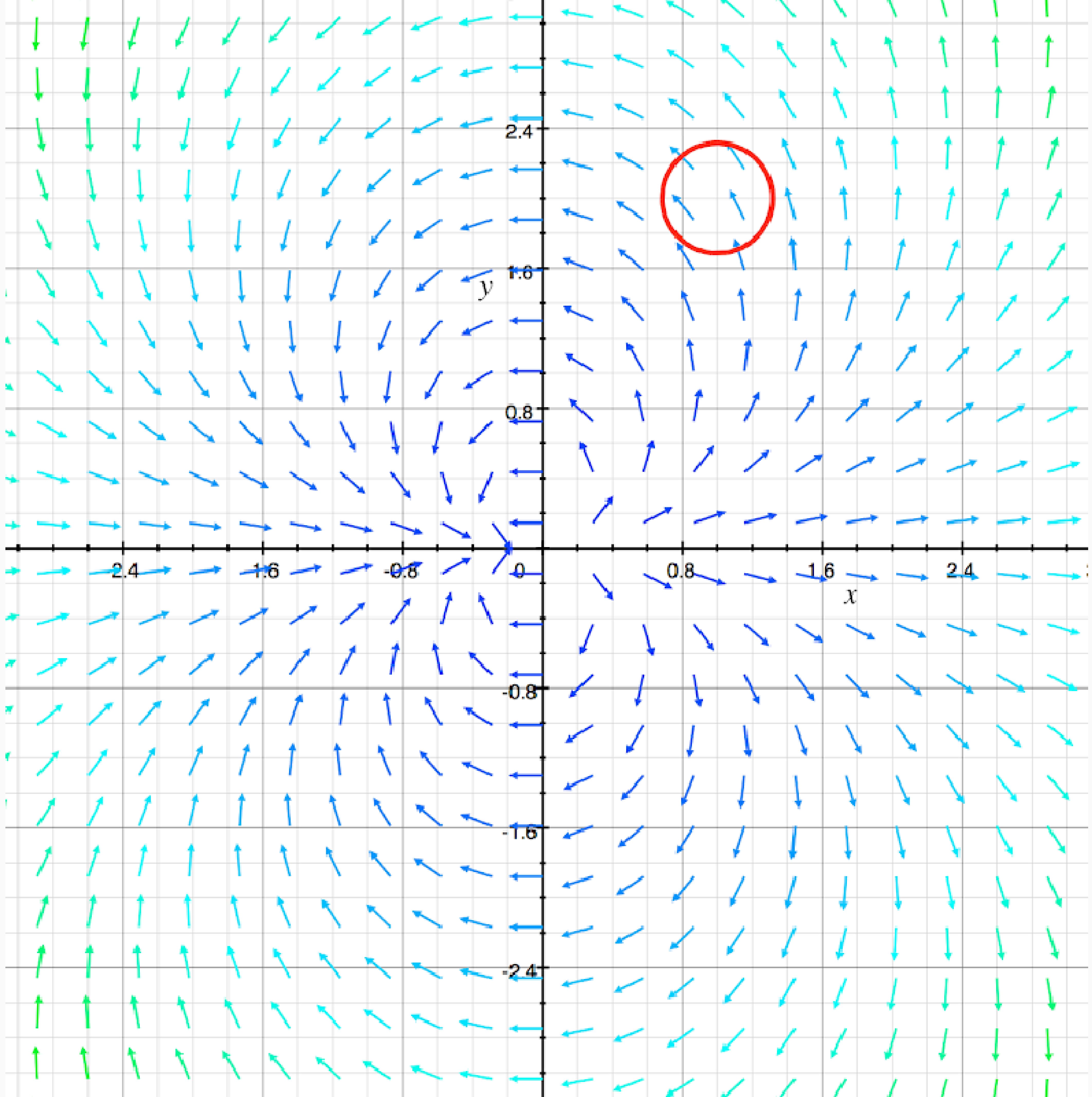


2D Div Theorem!

2D Div Theorem

Statement

$$\underbrace{\int F\cdot\hat{n}\,ds}_{\substack{\text{Line} \\ \int}} = \underbrace{\iint \nabla\cdot F\,dA}_{\text{Divergence}}$$

2D Div Theorem

Statement

$$\iint \underbrace{\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}}\, \partial \underline{\underline{A}}$$

2D Div Theorem

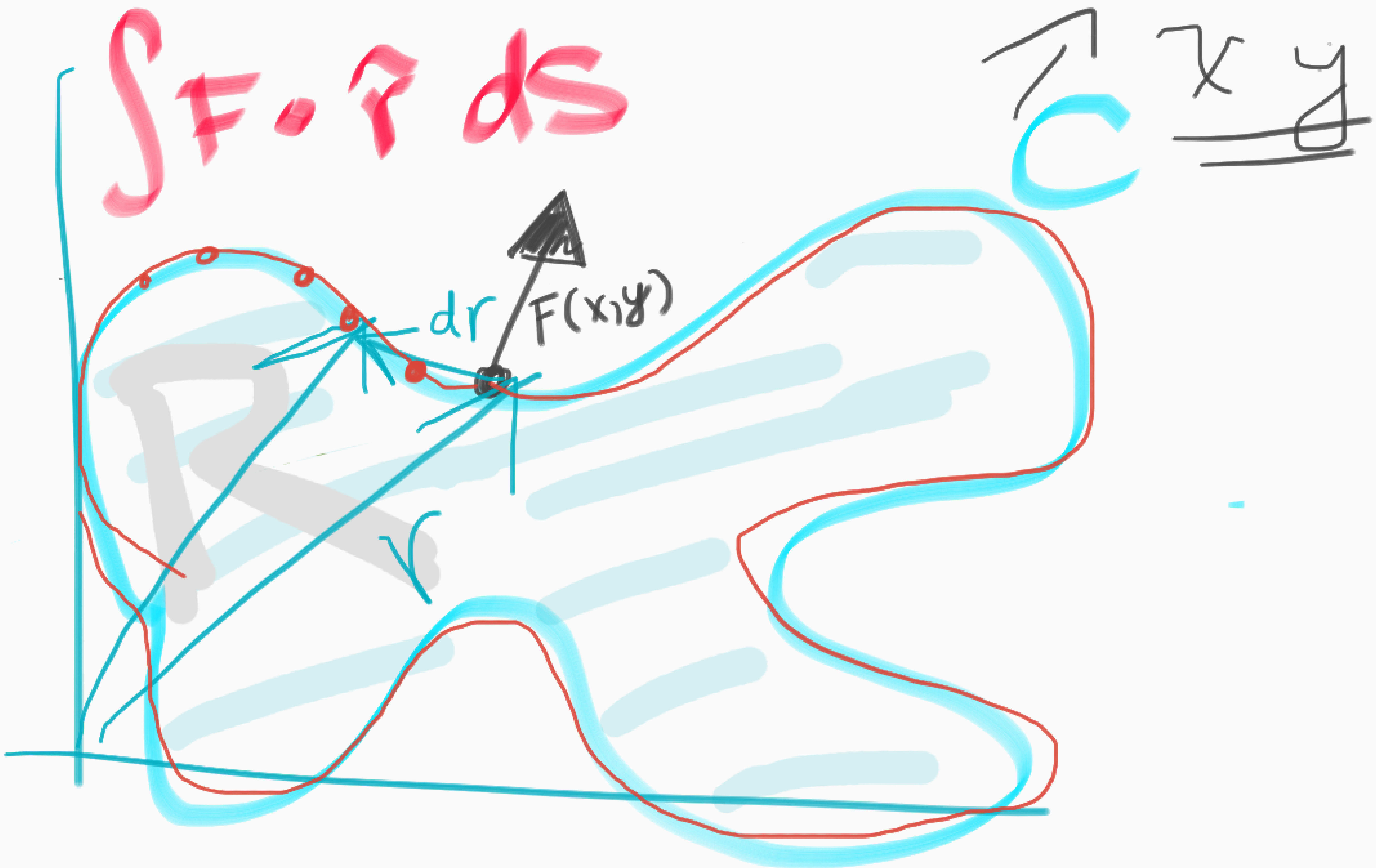
Flux

Div

2D Div Theorem

Flux

Div



2D Div Theorem

Flux

Div

$$\nabla \circ F = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}$$

local

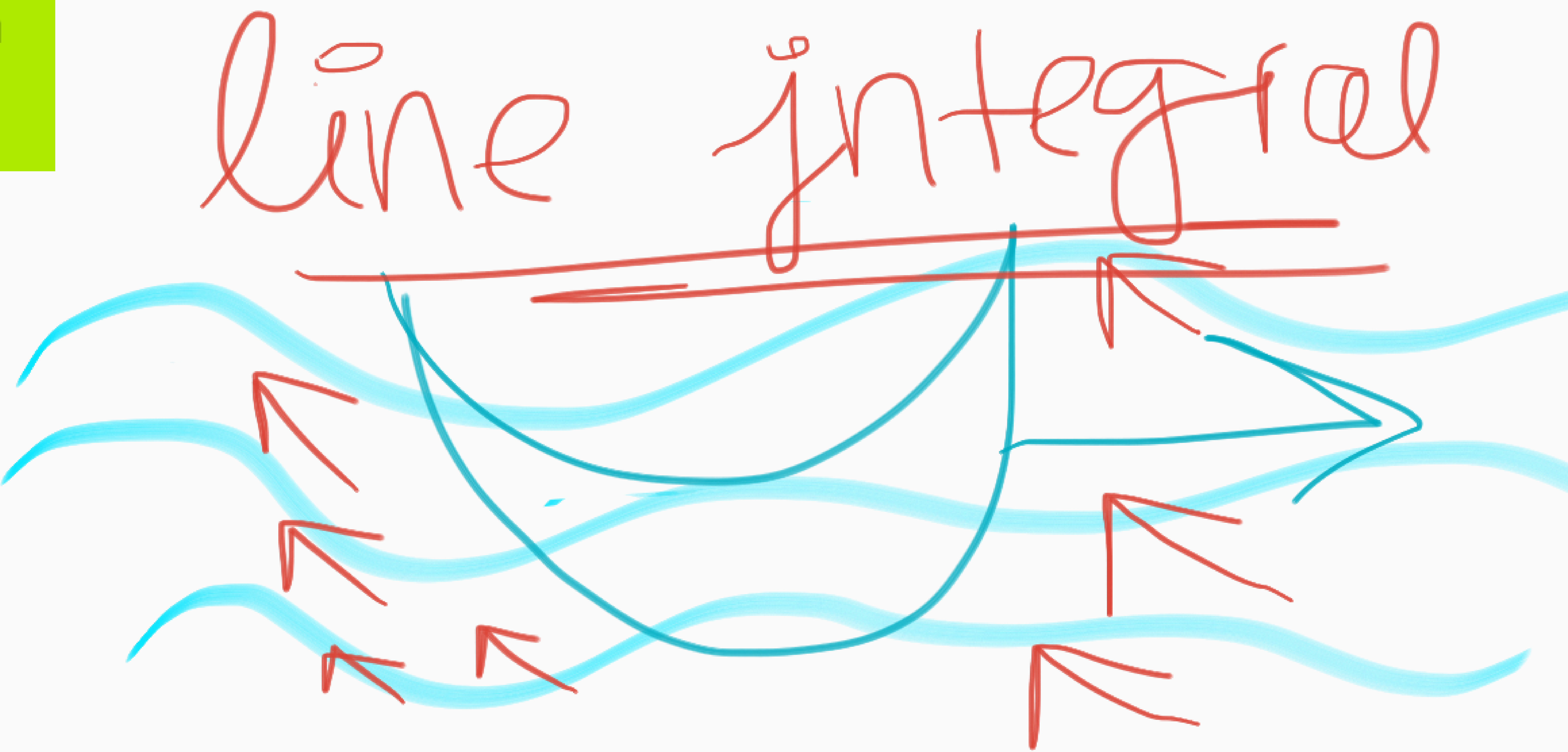
$$F = \begin{bmatrix} P(x,y) \\ Q(x,y) \end{bmatrix}$$

div F > 0 | div F < 0 | div F = 0

2D Div Theorem

Flux

Div



2D Div Theorem

Slice & Dice

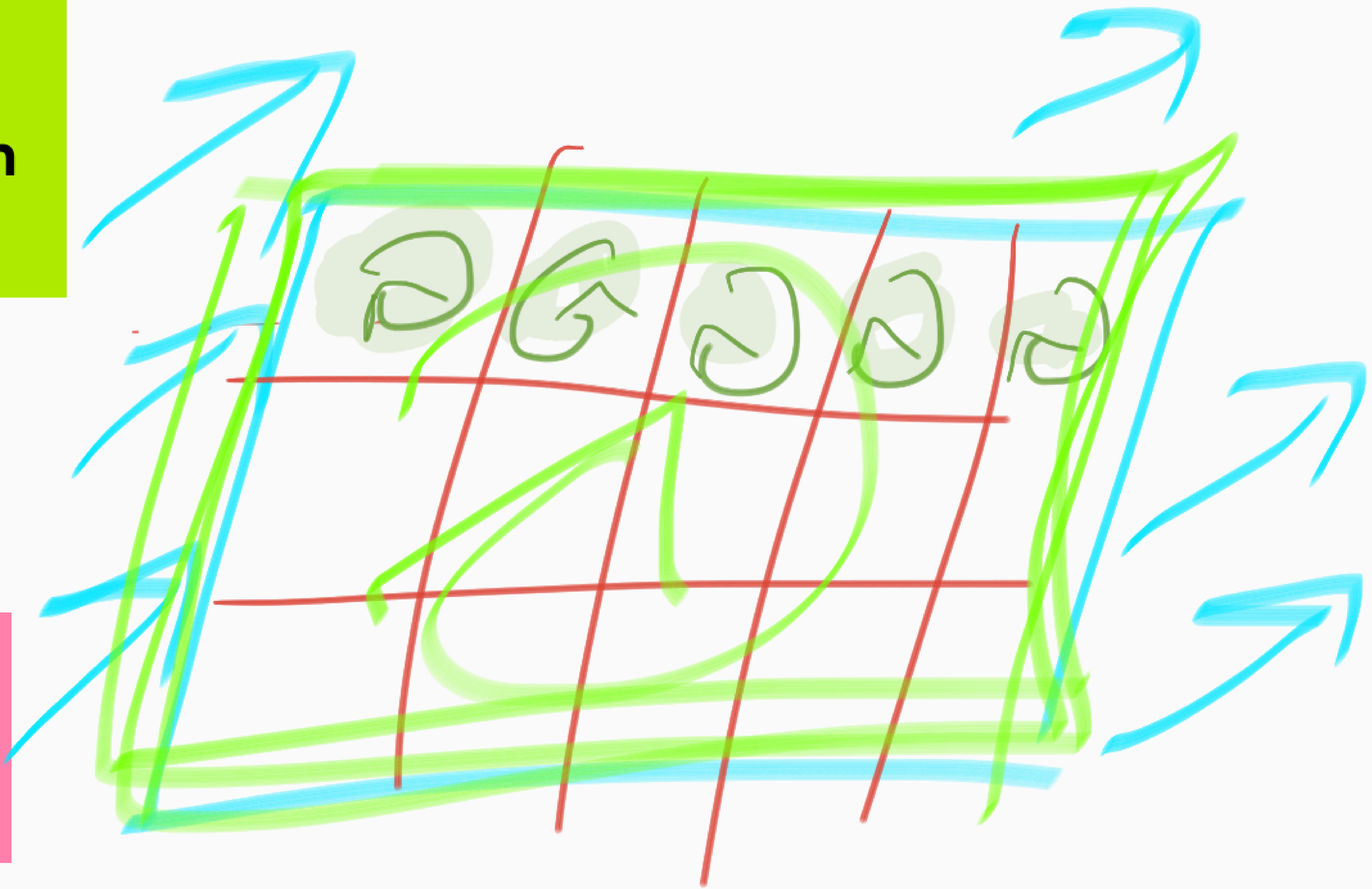
$$\iint_D \nabla \cdot F \, dA$$

$$\nabla_{C_1} \cdot F + \nabla_{C_2} \cdot F + \nabla_{C_3} \cdot F + \ldots = \sum_{i=1}^{n} \nabla_{C_n} \cdot \vec{F}$$

Green's Theorem

Curl

Circulation



Green's Theorem

Curl

Circulation

$$\oint_C F \cdot \hat{r}\, ds = \iint \operatorname{curl} F \, dA$$

Green's Theorem

Curl

Circulation

Line $\int$

Div

curl

mass

work

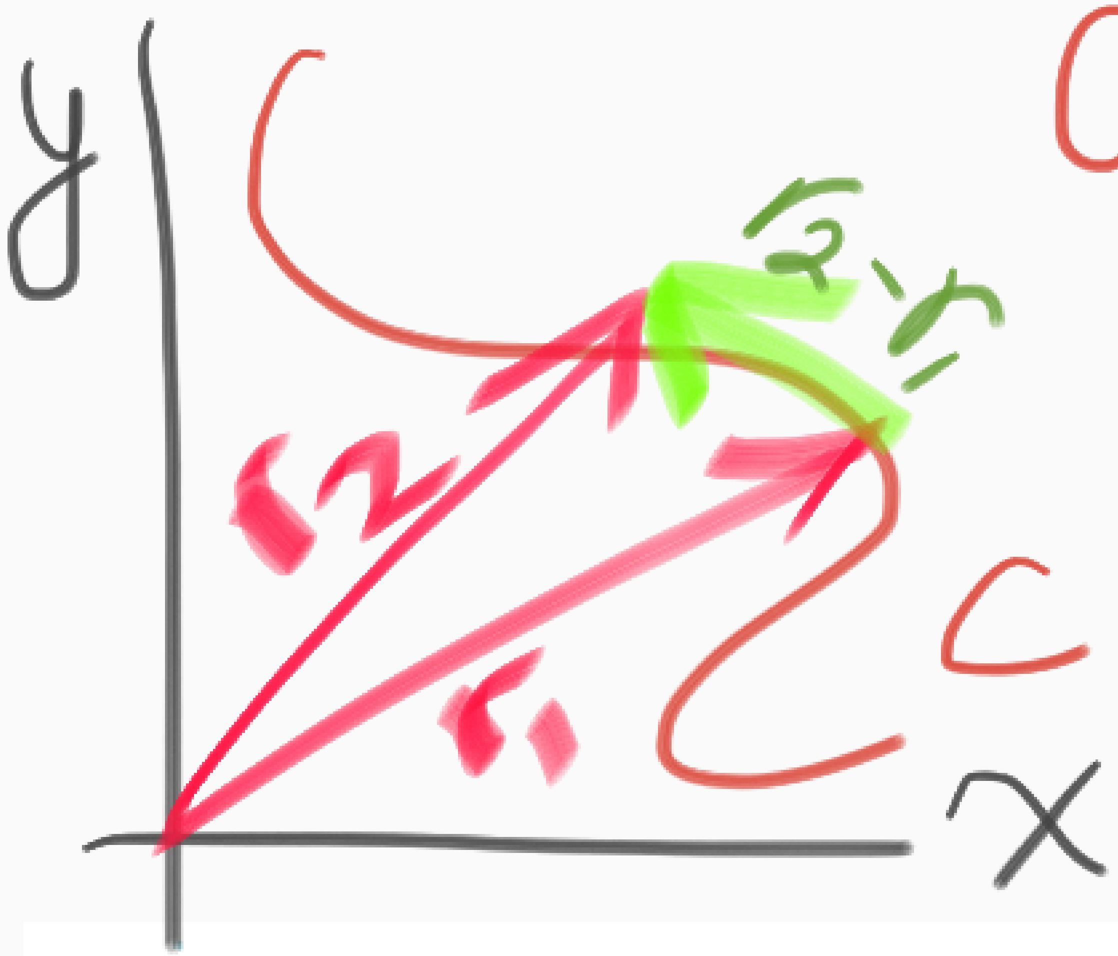
Proof of 2D Div Theorem

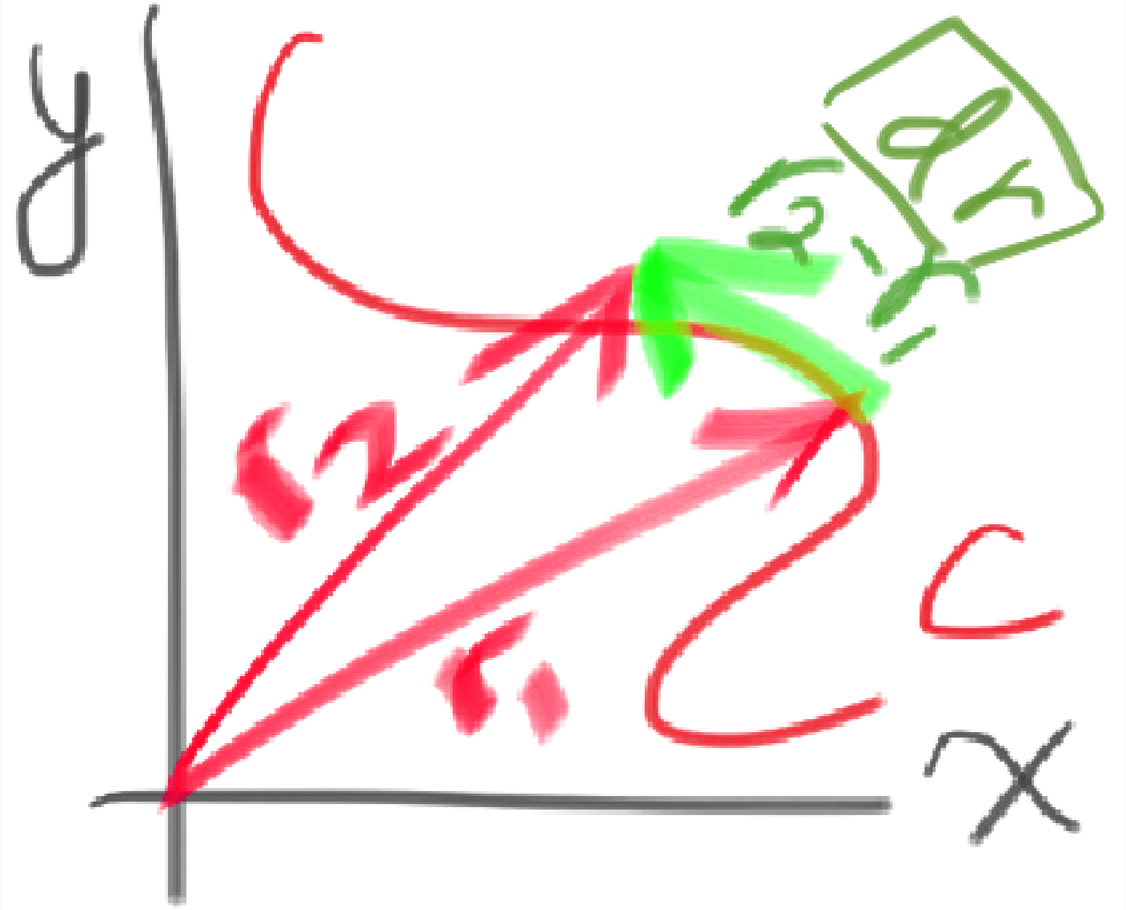
Finding Unit Vector

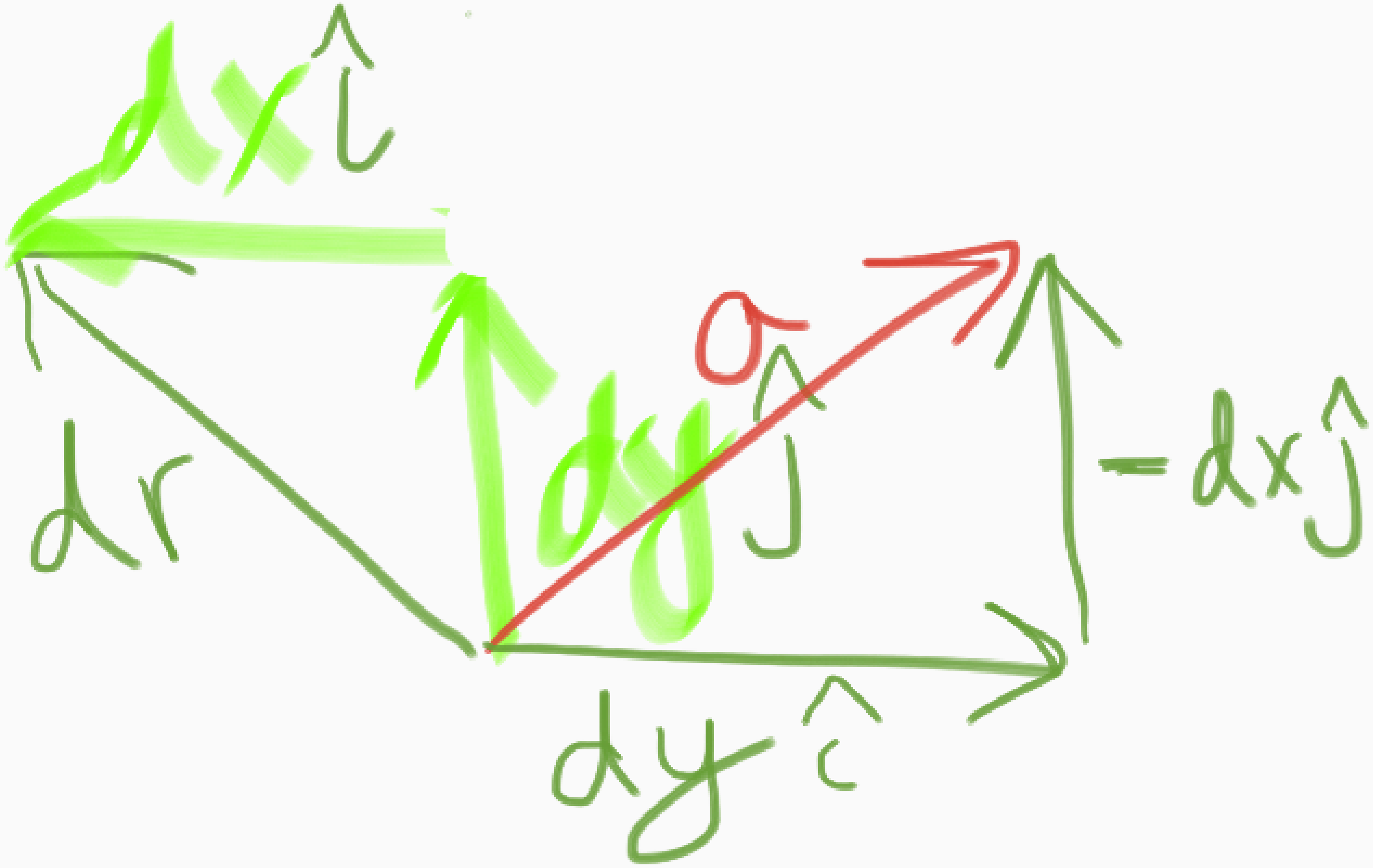
$$a = dy\,\hat{i} - dx\,\hat{j}$$

GOAL: Find $\hat{n}_{\perp}(x,y)$

as $r_1 \to r_2$, $(r_2 - r_1) \to dr$







Proof of 2D Div Theorem

Finding Unit Vector

$$a = dy\,\hat{i} - dx\,\hat{j}$$

$$|a| = \sqrt{dy^2 + dx^2} = \underline{\underline{ds}}$$

$$\hat{a} = \frac{dy\,\hat{i} - dx\,\hat{j}}{ds}$$

$$\boxed{= \hat{n}_{\perp}(x, y)}$$

Proof of 2D Div Theorem

Computing Line Integral

$$\oint_C F \cdot \hat{r}\, ds = \oint_C F \cdot \left(\frac{dy\,\hat{i} - dx\,\hat{j}}{\cancel{ds}} \right) \cancel{ds}$$

$$F = \begin{bmatrix} P(x,y) \\ Q(x,y) \end{bmatrix}$$

$$\oint_C P(x,y)\,dy - Q(x,y)\,dx$$

Proof of 2D Div Theorem

Computing Line Integral

$$\oint_C F \cdot \hat{r}\, ds = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, \underbrace{\partial x \partial y}_{\partial A}$$

$$\oiint_C \boxed{\frac{\partial P(x,y)}{\partial x} + \frac{\partial Q(x,y)}{\partial y}}\, \partial x \partial y$$

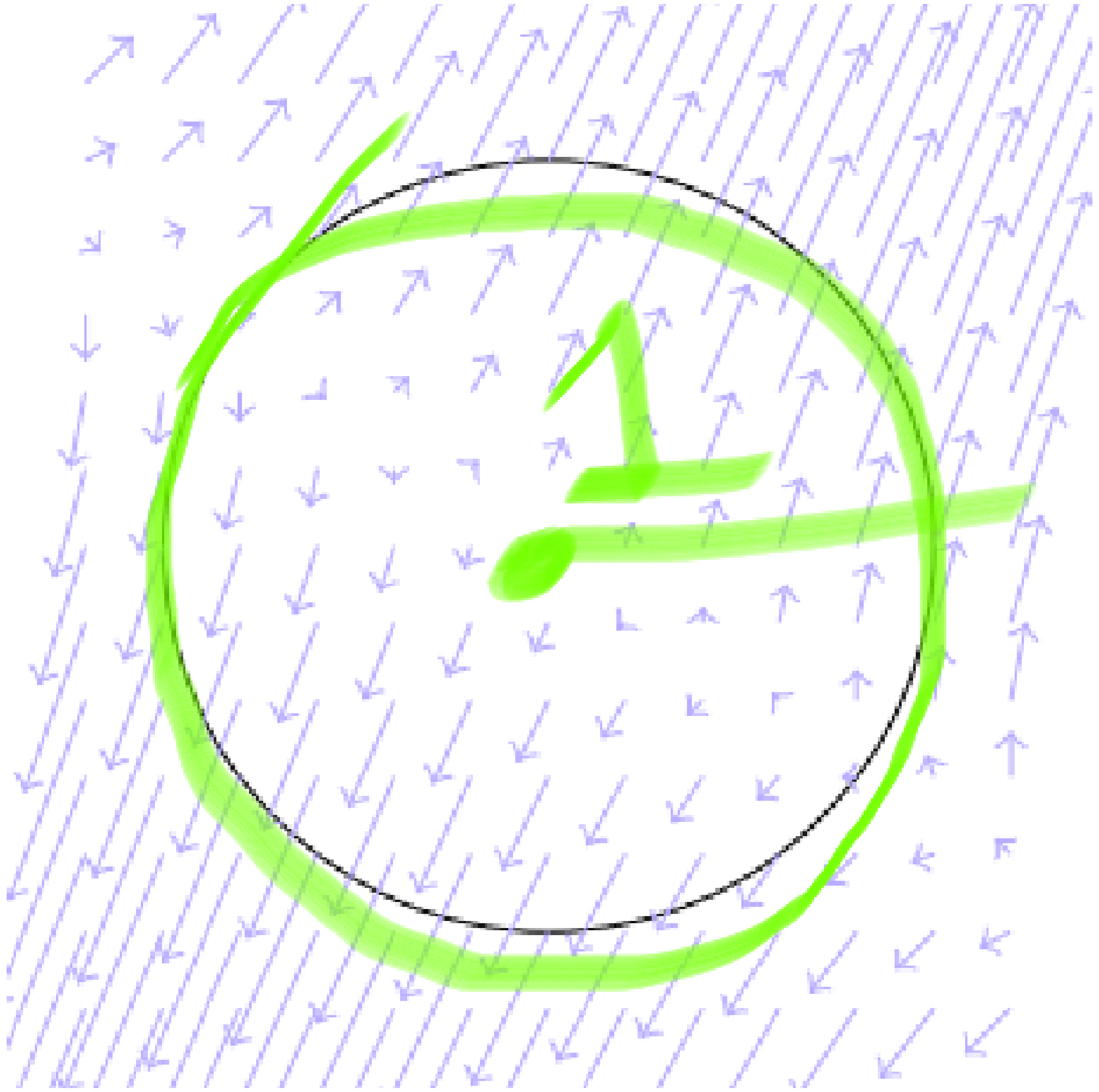
$$\boxed{\oiint_C \nabla \cdot F\, \partial A}$$

$$\oint_C P(x,y)\,dy - Q(x,y)\,dx$$

We will calculate the flow of the field

$$\boxed{\mathbf{F} = (x + 2y, 3x + 4y)}$$

out of the unit circle C.



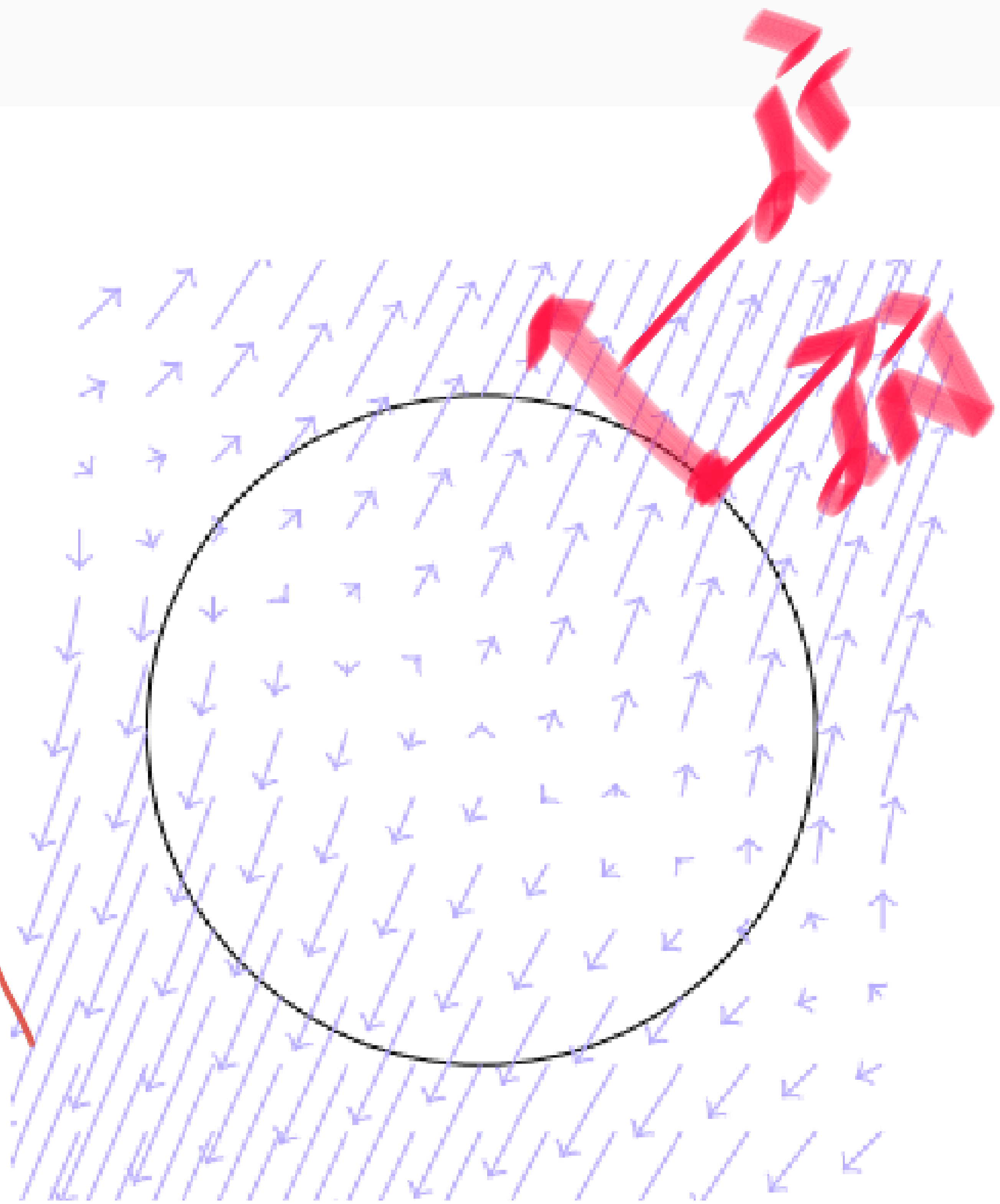
$$\int \vec{F} \circ \hat{r}\, ds$$

$$r(t) = \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix}$$

We will calculate the flow of the field

$$\mathbf{F} = (x + 2y, 3x + 4y)$$

out of the unit circle C.



$$r = \begin{array}{cc} \begin{bmatrix} \cos t \\ \sin t \end{bmatrix} & \begin{array}{l} - x \\ - y \end{array} \end{array}$$

$$dr = -\sin t \,\hat{i} + \cos t \,\hat{j}$$

$$dN = \cos t \,\hat{i} + \sin t \,\hat{j}$$

$$F = \cos t + 2\sin t \,\hat{i} + 3\cos t \cdot 4\sin t \,\hat{j}$$

$$dr = -\sin t\,\hat{i} + \cos t\,\hat{j}$$

$$dN = \cos t\,\hat{i} + \sin t\,\hat{j}$$

$$F = (\cos t + 2\sin t)\,\hat{i} + (3\cos t + 4\sin t)\,\hat{j}$$

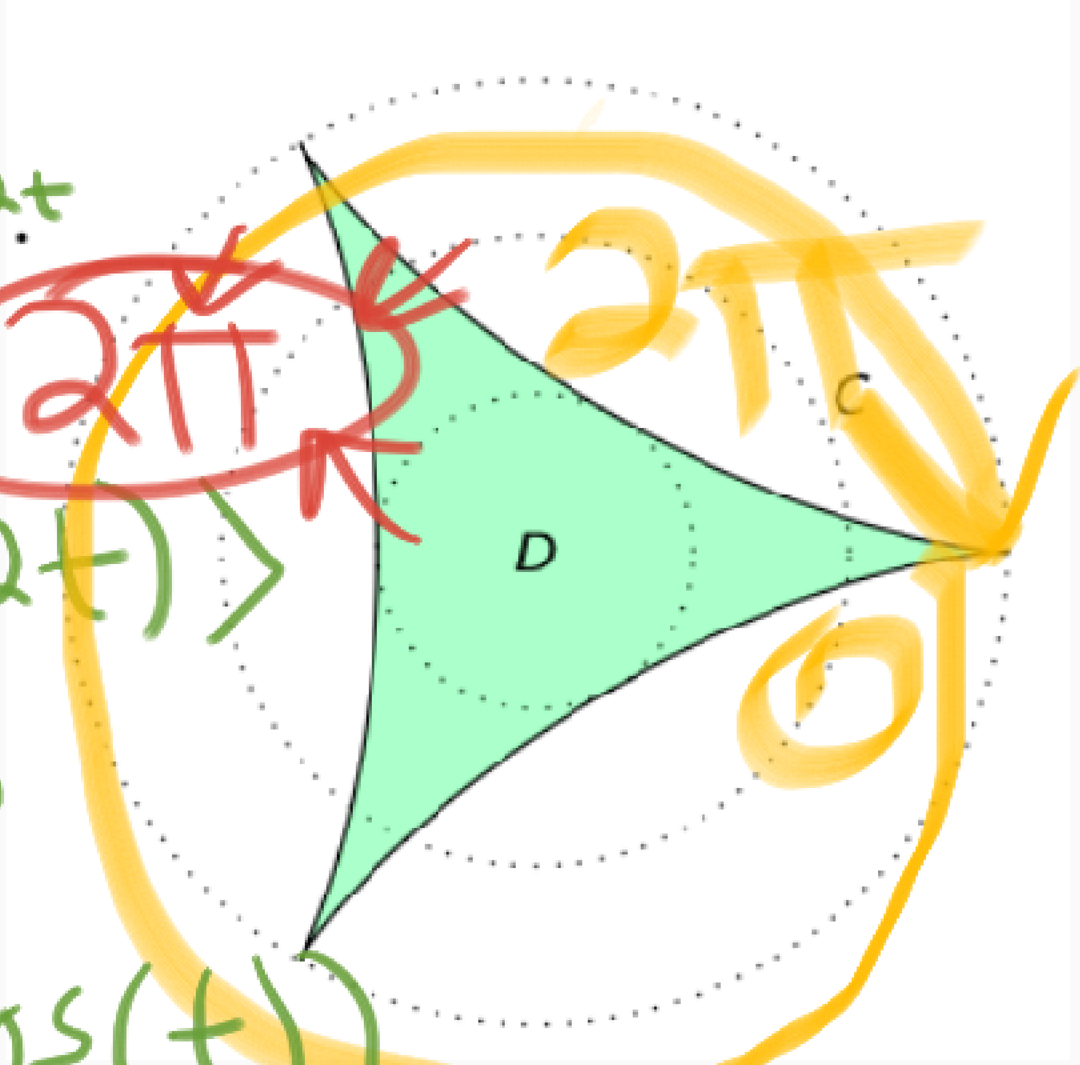
$$\int F \cdot d\underline{n}$$

$$F \cdot dn = (\cos^2 t + 2\sin t\cos t)\,\hat{i} + (3\sin t\cos t + 4\sin^2 t)\,\hat{j}$$

$$\int_0^{2\pi} \cos^2 t + 5\sin t\cos t + 4\sin^2(t)\; dt = 5\pi$$

The picture shows the deltoid curve C:

$x = 2\cos(t) + \cos(2t) \qquad y = 2\sin(t) - \sin(2t).$



$\int_0^{2\pi} 4\cos^2 t - 4\cos t\cos 2t + 2\cos t\cos 2t - 2\cos^2(2t) = 2\pi$

2π

0

$r = \langle 2\cos(t)\sin(t), 2\sin(t) - \sin(2t)\rangle$

$F = \langle 2\cos(t) + \cos(2t), 0\rangle$

$dr = \langle 2(-\sin(t)\sin(t) + \cos(t)\cos(t)),$
$2\cos(t) - 2\cos(2t)\rangle$

$dN = \langle 2\cos(t) - 2\cos(2t), -2(-\sin^2(t) + \cos^2 t)\rangle$

$F \cdot dN = \langle 4\cos^2 t - 4\cos(t)\cos(2t) + 2\cos(t)\cos(2t) - 2\cos^2(2t), 0\rangle$

Example

Find the flux of $\mathbf{F}(x, y) = \langle 2x + 2xy + y^2, x + y - y^2 \rangle$ across the circle $x^2 + y^2 = 4$.

$F \circ d\hat{n}$

$r = \langle 2\cos t, 2\sin t \rangle$

$r = \begin{bmatrix} 2\cos t \\ 2\sin t \end{bmatrix}$

$F = \langle 4\cos t + 8\sin t\cos t + 4\sin^2 t,$
$2\cos t + 2\sin t - 4\sin^2 t \rangle$

$dr = \langle -2\sin t, 2\cos t \rangle$

$F \circ d\hat{N} =$

$dN = \langle 2\cos t, 2\sin t \rangle$

$\int_0^{2\pi} 8\cos^2 t + 16\cos^2 t\sin t + 8\sin^2 t\cos t + 4\sin t\cos t + 4\sin^2 t - 8\sin^3 t$